NASA Contractor Report 189575

ICASE Report No. 91-85

ICASE

CONVERGENCE OF CONTROLLERS DESIGNED USING STATE SPACE METHODS

K. A. Morris

NASA

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665-5225

Convergence of Controllers Designed Using State Space Methods

K. A. Morris[1]

Department of Applied Mathematics

University of Waterloo

Waterloo, Ontario N2L 3G1

ABSTRACT

In this paper convergence of finite-dimensional controllers for infinite-dimensional systems designed using approximations is examined. Stable coprime factorization theory is used to show that under the standard assumptions of uniform stabilizability/detectability, the controllers stabilize the original system for large enough model order. The controllers converge uniformly to an infinite-dimensional controller, as does the closed loop response.

1 Introduction

We consider *semigroup control systems* on a Hilbert space X:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

$$y(t) = Cx(t),$$

$$x(0) = x_0, \qquad x_0 \in D(A) \subset X.$$

The operator A generates a strongly continuous semigroup of operators $S(t)$ on X so A is closed with domain $\mathcal{D}(A)$ dense in X. We assume bounded control and observation, and that the input and output spaces are finite-dimensional: $B \in \mathcal{B}(R^m, X), C \in \mathcal{B}(X, R^p)$. This control system will often be abbreviated (A, B, C). For control functions in $\mathcal{L}_2(0, \infty; R^m)$ we interpret the solution of (1) in the mild sense:

$$y(t) = CS(t)x_0 + C \int_0^t S(t - \tau)Bu(\tau)d\tau.$$

Further details can be found in, for instance [14].

The equations (1) are a model for a number of control problems, including those where the system dynamics are described by partial differential equations and hereditary differential systems. A closed form solution can be computed only in the simplest of situations. In general, it is necessary to use an numerical approximation to the semigroup control system (1) in order to simulate the response of the system. This approximation will typically be a system of n ordinary differential equations, which we write as

$$\dot{x}(t) = A_n x(t) + B_n u(t), \tag{2}$$

$$y(t) = C_n x(t),$$

$$x(0) = x_{0n}.$$

This approximation is also used to compute controllers for the original system (1). Further details on the approximation scheme will be presented in a subsequent section. The above finite-dimensional system will often be abbreviated as (A_n, B_n, C_n).

In this paper, we are concerned with finite-dimensional controllers of the form

$$\dot{z}(t) = (A_n - B_n K_n)z(t) + F_n(y(t) - C_n z(t)) \tag{3}$$

$$u_n(t) = K_n z(t).$$

There are a number of convergence questions associated with this approach:

- Does the controller (3) stabilize (1) for large enough n?

- Do the controllers (3) converge to an infinite-dimensional controller?

- What relation does the closed loop performance of the controller (3) with (A_n, B_n, C_n) have to its implementation with the original system (A, B, C)?

Convergence in the *graph topology* [18, 19] of the approximating systems to the original system is necessary for the validity of an approximation scheme as a basis for controller design. Convergence in the graph topology is equivalent to convergence of a sequence of coprime factors (N_n, D_n) for the approximations to a coprime factorization for the original system. Without such convergence it cannot be concluded that a controller which stabilizes (A, B, C) also stabilizes (A_n, B_n, C_n) for sufficiently large n, or that the closed loop responses converge. In fact, in the absence of convergence in the graph topology, the set of controllers which stabilize both the infinite-dimensional system (A, B, C) and the approximations (A_n, B_n, C_n) may be empty, even for large n.

In this paper this approach is used to obtain complete answers to the above questions for a wide class of systems. A complete answer to the problem of convergence of LQG type controllers is given. Although the graph topology is the fundamental topology underlying these convergence questions, it is not used explicitly in this paper. All results are derived directly using only theory on Bezout, or stable coprime factors. First, some background material on stable coprime factorizations is given.

2   Stable Coprime Factorizations

Suppose a system of the form (1) with $x_0 = 0$ maps inputs in $\mathcal{L}_2(0, \infty, R^m)$ to outputs in $\mathcal{L}_2(0, \infty, R^p)$, and that furthermore, there is a maximum ratio, the $\mathcal{L}_2$-gain between the norm of the output and the norm of the input:

$$\| y \|_{\mathcal{L}_2(0,\infty,R^p)} \leq \gamma \| u \|_{\mathcal{L}_2(0,\infty,R^m)}.$$

Then the system is said to be $\mathcal{L}_2$-stable, and by the Paley-Weiner Theorem, the Laplace transform of such a system is a matrix with entries in $\mathcal{H}_\infty$. Here $\mathcal{H}_\infty$ indicates the Hardy space of functions $G(s)$ which are analytic in the right-half plane $Re(s) > 0$ and for which

$$\sup_\omega \sup_{x>0} |G(x + j\omega)| < \infty.$$

The norm of a function in $\mathcal{H}_\infty$ is

$$\| G \|_\infty = \sup_\omega \sup_{x>0} |G(x + j\omega)|.$$

2

We denote matrices with entries in $\mathcal{H}_\infty$ by $M(\mathcal{H}_\infty)$. The norm of a function in $M(\mathcal{H}_\infty)$ is the induced matrix norm

$$\| G \|_\infty = \sup_{x>0} \sup_\omega \bar{\sigma}\left[G(x+j\omega)\right]$$

where $\bar{\sigma}$ denotes the largest singular value.

The Laplace transform of a linear time-invariant system such as (1) is called its transfer function and is given by $CR(s;A)B$ where $R(s;A)$ indicates the resolvant of A. Suppose G is the transfer function of a given system, for which we wish to design a controller with transfer function H, of compatible dimensions, arranged in the feedback configuration shown in Figure 1.
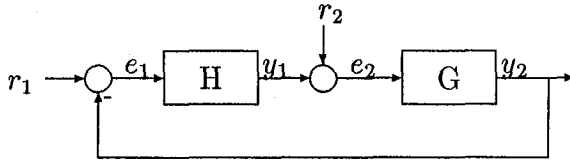

Figure 1. Feedback System

The 2×2 transfer matrix $\Delta(G,H)$ which maps the pair (r_1, r_2) into the pair (e_1, e_2) is given by

$$\Delta(G,H) = \begin{bmatrix} (I+GH)^{-1} & -G(I+HG)^{-1} \\ H(I+GH)^{-1} & (I+HG)^{-1} \end{bmatrix}.$$

The feedback system, or alternatively the pair (G,H), is said to be *externally stable* if each of the four elements in the above matrix belongs to the set **S** of stable transfer functions. We could define stability in terms of the transfer matrix from (r_1, r_2) to (y_1, y_2); both notions of stability are equivalent [19]. Definition of **S** depends upon the application. Thus the closed loop system is $\mathcal{L}_2$-stable if and only if all four elements belong to $M(\mathcal{H}_\infty)$. The set of all plants which stabilize G is written $S(G)$:

$$S(G) = \{ H \mid \Delta(G,H) \in M(\mathcal{H}_\infty) \}.$$

Note that the present definition of stability is symmetric in G and H. Thus G stabilizes H if and only if H stabilizes G.

For the common situation where the system is already stable and an aim of controller design is to improve the settling time of the system, we specify a real number $\sigma > 0$ which is the minimum acceptable stability margin. Then a system is said to be input/output

3

σ -stable if its shifted transfer function $G(s - \sigma)$ is in $M(\mathcal{H}_\infty)$. Equivalently, we replace $\mathcal{H}_\infty$ by the algebra $H_{\infty\sigma}$ of functions which are analytic in the right half plane $Re(s) > -\sigma$, and for which

$$\sup_{\omega} \sup_{x > -\sigma} |G(x + j\omega)| < \infty$$

with corresponding norm.

Much of modern control theory is concerned with *coprime factorizations* of systems. The transfer function of a possibly unstable system G is written as the ratio of two coprime stable systems. For the case of $\mathcal{L}_2$-stability, the transfer function of a system is written as $G = ND^{-1}$ where $N, D \in M(\mathcal{H}_\infty)$ and there exists $X, Y \in M(\mathcal{H}_\infty)$ with

$$X(s)N(s) + Y(s)D(s) = I, \qquad Re(s) \geq 0. \tag{4}$$

(N, D) is called a *right coprime factorization* (r.c.f.) for G. *Left coprime factorizations* (l.c.f.'s) are defined similarly. $(\tilde{N}, \tilde{D})$ is a l.c.f for G if $G = \tilde{D}^{-1}\tilde{N}$ where $\tilde{N}, \tilde{D} \in M(\mathcal{H}_\infty)$ and there exists $\tilde{X}, \tilde{Y} \in M(\mathcal{H}_\infty)$ with

$$\tilde{N}(s)\tilde{X}(s) + \tilde{D}(s)\tilde{Y}(s) = I, \qquad Re(s) \geq 0. \tag{5}$$

Every system which is described by a system of linear time-invariant ordinary differential equations has both a left- and a right-coprime factorization. (This is a consequence of the fact that the transfer functions of such plants are composed of rational functions.) Furthermore, the set of all stabilizing controllers for such plants may be described in terms of the *Youla parameterization*; a controller H externally stabilizes G if and only if it can be written

$$H = (Y - R\tilde{N})^{-1}(X + R\tilde{D}), \quad |Y - R\tilde{N}| \neq 0, R \in M(\mathcal{H}_\infty) \tag{6}$$

where $X, Y, \tilde{N}, \tilde{D}$ are as defined in (4), (5). The set of all stabilizing controllers for a given system G are parameterized by R , as R ranges over all stable systems. In other words,

$$S(G) = \{(Y - R\tilde{N})^{-1}(X + R\tilde{D}), \quad |Y - R\tilde{N}| \neq 0, R \in M(\mathcal{H}_\infty)\}. \tag{7}$$

The above formulation (7) is in terms of left coprime factors of the stabilizing controllers. The same family, $S(G)$ may also be written in terms of right coprime factors:

$$S(G) = \{(\tilde{X} + DQ)(\tilde{Y} - NQ)^{-1}, \quad |\tilde{Y} - NQ| \neq 0, Q \in M(\mathcal{H}_\infty)\}. \tag{8}$$

Unfortunately, more general systems, which do not have rational transfer functions, do not necessarily have either a left- or a right-coprime factorization. However, any system with a transfer function which can be written as a fraction $ND^{-1}, N, D \in M(\mathcal{H}_\infty)$, is stabilizable

4

if and only if it has right (and left) coprime factorizations [16] . For systems of the type (1), the theory is more complete. First, some definitions are required.

Definition 1.1: The C_o-semigroup $T(t)$ is stable if there exist constants M and $\alpha > 0$ such that $\|T(t)\| \leq Me^{-\alpha t}$ for all $t \geq 0$.

Definition 1.2 A semigroup control system (A, B, C) is said to be *internally stable* if the semigroup generated by A is stable according to Definition 1.1 .

Definition 1.3: The pair (A, B) is *stabilizable* if there exists a bounded linear operator $K : X \to R^m$ such that $A - BK$ generates a stable semigroup.

Definition 1.4: The pair (A, C) is *detectable* if there exists a bounded linear operator $F : R^p \to X$ such that $A - FC$ generates a stable semigroup.

Definition 1.5: The system (A, B, C) is jointly *stabilizable/detectable* if (A, B) is stabilizable and (A, C) is detectable.

The extensions to σ-stable, σ-stabilizable etc. are straightforward.

Jointly stabilizable/detectable systems (A, B, C) have both left and right coprime factorizations [5]. Let K be such that $A - BK$ generates a stable semigroup and F such that $A - FC$ generates a stable semigroup. Define

$$N(s) = CR(s, A - BK)B, \tag{9}$$

$$D(s) = I - KR(s, A - BK)B, \tag{10}$$

$$X(s) = KR(s, A - FC)F,$$

and

$$Y(s) = I + KR(s, A - FC)B,$$

$$XN + YD = I.$$

(N, D) is a r.c.f for the system *i.e.* $CR(s; A)B = ND^{-1}$ [10] and the Youla parameterization describes all stabilizing controllers for these systems. A l.c.f. can be defined similarly, using an operator F such that $A - FC$ generates a stable semigroup . Define

$$\tilde{N}(s) = CR(s, A - FC)B$$

$$\tilde{D}(s) = I - CR(s, A - FC)F.$$

Then, $G(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$ and

$$\tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I$$

5

for some $\tilde{X}, \tilde{Y} \in M(\mathcal{H}_\infty)$.

It is easy to show, using the Hille-Yosida Theorem, that internally stable control systems (1) are externally stable. We are also assured that external stability implies internal exponential stability:

Theorem 2.1 *[10] A jointly stabilizable/detectable semigroup control system is internally stable if and only if it is externally stable.*

This equivalence justifies the use of controller design techniques based on system input/output behaviour for infinite-dimensional systems of the form (1). The discussion above is summarized in the following theorem.

Theorem 2.2 *Assume (A, B, C) is jointly stabilizable/detectable and let G be the transfer function of the system. We say that a controller with transfer function H stabilizes G if the closed loop matrix $\Delta(G, H) \in M(\mathcal{H}_\infty)$. Then H stabilizes the system, if and only if, its transfer function can be written using the Youla parameterization (7) (or (8)).*

Also, if the state-space representations of the systems G and H are each stabilizable and detectable, then H stabilizes G, according to the definition at the beginning of this section, if and only if the closed loop is also internally stable.

Theorem 2.3 *[10] Assume (A, B, C) is jointly stabilizable/detectable and that H is a controller with a jointly stabilizable/detectable realization (A_c, B_c, C_c). The closed loop system is input/output stable if and only if it is internally stable.*

The following is by now well-known, and several proofs exist. One version, which uses coprime factorizations, and which first appeared in [13] is given here.

Theorem 2.4 *Every jointly stabilizable/detectable system (1) is stabilizable by a finite-dimensional controller.*

Proof: Both the stable factors N, D have well-defined limits at infinity:

$$\lim_{\substack{|s| \to \infty \\ Re(s) \geq 0}} \overline{\sigma}(N(s)) = 0, \qquad \lim_{\substack{|s| \to \infty \\ Re(s) \geq 0}} \overline{\sigma}(D(s)) = 1.$$

It follows (Mergelyan's Theorem) that each factor can be approximated by a rational element of $M(\mathcal{H}_\infty)$: for any $\epsilon > 0$ we can find rational $N_n, D_n \in M(\mathcal{H}_\infty)$, with

$$\| N(s) - N_n(s) \|_\infty < \epsilon, \; \| D(s) - D_n(s) \|_\infty < \epsilon.$$

For sufficiently small ϵ, (N_n, D_n) is also a coprime pair. Let (X, Y) be a l.c.f of any finite-dimensional controller which stabilizes $N_n D_n{}^{-1}$ with $X N_n + Y D_n = I$. Then, if ϵ is small enough, $Y^{-1}X$ also stabilizes the infinite-dimensional system since $XN + YD = U$ where U has an inverse in $M(\mathcal{H}_\infty)$. Rewriting, we have $X(NU^{-1}) + Y(DU^{-1}) = I$. Since (NU^{-1}, DU^{-1}) is a r.c.f. for the infinite-dimensional system (1), it follows that every jointly stabilizable/detectable control system (1) is stabilizable by a finite-dimensional controller. $\square$

3 Approximation Scheme

In this paper we are interested in the case where a finite-dimensional controller is designed using a numerical approximation to the original system.

Suppose we have a sequence of finite-dimensional subspaces $X_n \subset X$ where the norm on X_n is that inherited from X. Define $P_n x$ as the orthogonal projection of $x \in X$ onto the finite-dimensional subspace X_n. We assume the following:

(A1) The projection operators P_n converge strongly to the identity on the Hilbert space X. That is, for all $x \in X$
$$\lim_{n \to \infty} \| P_n x - x \| = 0.$$

For each X_n, the approximating system is (A_n, B_n, C_n):

$$B_n := P_n B, \qquad C_n := C|_{X_n},$$

and A_n is an approximation to A which satisfies the two assumptions (A2) and (A3) below. Note that the operators A_n, B_n, C_n and the semigroup $T_n(t)$ generated by A_n are operators on X_n.

A *core* C of a closed operator A is a linear space contained in the domain of A with the property that the set of elements (x, Ax), $x \in$ C is dense in the graph $\mathcal{G}(A)$ of the operator A ([11], pg. 166).

(A2) We assume that there exists a core C for A such that

$$\lim_{n \to \infty} \quad \| P_n A x - A_n P_n x \| = 0, \quad \text{for all } x \in \text{C}. \tag{11}$$

Such an approximation scheme is said to be *consistent*.

(A3) We will further assume that the semigroups $T_n(t)$ generated by A_n are uniformly bounded, that is, there exist real numbers $N, M \geq 1$ and k such that

$$\|T_n(t)\| \leq Me^{kt} \text{ for all } n \geq N. \tag{12}$$

(Uniform boundedness of the approximate semigroups is generally referred to as "stability" in the numerical analysis literature.)

Consistency and uniform boundedness are sufficient for *convergence* of the approximation [11, 14] *i.e.*, for all $\varepsilon > 0, t > 0$ and for all $x \in X$ there exists N such that

$$\|P_n T(\tau)x - T_n(\tau)P_n x\| < \varepsilon \text{ for all } \tau \in [0, t] \text{ and } n > N. \tag{13}$$

Assumptions (A1)-(A3) are satisfied by typical approximation methods. They are sufficient to ensure that the *open loop* response of the systems G_n approximate the response of G. (A1) is of course redundant in that it is implied by (13). Assumption (A2) may be replaced by a requirement of strong convergence of the resolvants. (Trotter-Kato Approximation Theorem *eg.* [14]). However, these assumptions are not sufficient to ensure convergence of the closed loop response. An example of a scheme which approximates the open loop response of a system but is not satisfactory as a basis for controller design is given in [4].

The appropriate topology in which to establish convergence of the approximations is the *graph topology*. The importance of the graph topology in controller design is due to the following result: A family of plants G_n can be robustly stabilized by a compensator H which stabilizes a nominal plant G if and only if G_n converges to G in the graph topology. Furthermore, in this case, the closed loop response of the feedback pair $\Delta(G_n, H)$ converges to $\Delta(G, H)$. In order to obtain convergence of the closed loop response $\Delta(G_n, H_n)$ to $\Delta(G, H)$ for some $H \in S(G)$, H_n must also converge to H in the graph topology. The results given in this paper can be shown directly using the coprime factorization theory outlined in the previous section. The graph topology is not directly discussed further in this paper, although it is the topology underlying the convergence results. The interested reader is referred to [19] for an introduction, [20] for extensions to general algebras, [12] for its application to approximation of semigroup control systems, and [6] for robust stabizablity under perturbations in the coprime factors.

Suppose that the approximation scheme also satisfies assumptions of uniform stabilizability, and uniform detectability:

(A4) If the original system is stabilizable, then the approximations are uniformly stabilizable: there exists a uniformly bounded sequence of operators $\{K_n\}$ such that for

sufficiently large N, the semigroups $S_n(t)$ generated by $A_n - B_n K_n$ are uniformly bounded by $Me^{-\alpha t}$ for some $M > 0, \alpha > 0$ and all $n > N$.

(A5) If the original system is detectable, then the approximations are uniformly detectable: there exists a uniformly bounded sequence of operators $\{F_n\}$ such that for sufficiently large N, the semigroups $S_n(t)$ generated by $A_n - F_n C_n$ are uniformly bounded by $Me^{-\alpha t}$ for some $M > 0, \alpha > 0$ and all $n > N$.

Common approximation schemes for several important classes of systems which satisfy these assumptions are given below. Not only do both approximation schemes discussed below satisfy assumptions (A1)-(A5), there exists a sequence of operators K_n which satisfy assumption (A4) and which converge strongly to an operator $K \in \mathcal{B}(X, U)$. Similar convergence exists for a sequence of operators F_n which satisfy assumption (A5).

3.1 Hereditary Systems

Consider the delay functional differential equation

$$\dot{x}(t) = \int_{-h}^{0} d\eta(\tau) x(t + \tau) + Bu(t), \qquad t \geq 0 \qquad (14)$$

$$y(t) = Cx(t),$$

$$x(0) = x_0, \ x(\tau) = \phi(\tau), -h \leq \tau < 0$$

where $x(t) \in R^n, y(t) \in R^p, u(t) \in R^m$ and B and C are matrices of appropriate dimension. Also, $\eta(\tau)$ is function of bounded variation taking values in $R^{n \times n}$ with $\eta(\tau) = 0$ and $\eta(\tau)$ is left continuous for $-h \leq \tau \leq 0$. Defining the state-space $X = R^n \times \mathcal{L}_2(-h, 0; R^n)$, the hereditary system (14) can be formulated as a control system of the form (1) [2].

Define the finite-dimensional subspaces of X to be

$$X_N = \left\{ (\phi_1, \phi_2) \in X; \ \phi_2(\tau) = z_j, \ \frac{-j}{N}h \leq \tau < \frac{-(j-1)}{N}h, \ j = 1, ... N \text{ where } z_j \in R^n \right\}.$$

The finite-dimensional Galerkin approximation to (14) derived using these subspaces satisfies assumptions (A1)-(A3) [2, 15]. This scheme is known as the averaging approximation to (14).

In [8] it is shown that this scheme is uniformly stabilizable and detectable: assumptions (A4) and (A5) are satisfied . It is also shown that the stronger version of (A4) and (A5) holds. For a bounded operator K such that $A - BK$ generates a stable semigroup, we can choose $K_n = K P_n$, K_n satisfies (A4) and $\lim_{n \to \infty} \| K_n - K \| = 0$. Similarly, if (14) is detectable, we can choose a sequence $F_n \to F$ where $A - FC$ generates a stable semigroup and F_n is a sequence which satisfies (A5).

9

3.2 Sectorial Operators

Define A through a continuous sesquilinear form $a : V \times V \mapsto \mathcal{C}$

$$\langle -A\phi, \psi \rangle = a(\phi, \psi), \quad \phi \in \mathcal{D}(A), \psi \in V. \tag{15}$$

where $\langle \cdot, \cdot \rangle$ indicates the inner product on X, V is a Hilbert space , $\mathcal{D}(A) \subset V$ and V is densely and continuously in X. In order to avoid confusion with the norm on X, the norm on V will be indicated by $\| \ \|_V$. Identify X with its dual so that $V \hookrightarrow X = X' \hookrightarrow V'$. We assume that in addition to (15), $a(,)$ satisfies Harding's inequality: there exists $k, c > 0$ such that

$$a(u, u) + k\langle u, u \rangle \geq c\| u \|_V^2. \tag{16}$$

The inequalities (15) and (16) guarantee that A generates an analytic semigroup with bound

$$\| T(t) \| \leq e^{kt}.$$

Further details may be found in [17]. Assume that the approximating subspaces X_n satisfy a V-approximation property: for all $x \in V$ there exists a sequence $x_n \in X_n$ with

$$\text{(H1)} \qquad \lim_{n \to \infty} \| x_n - x \|_V = 0.$$

It is shown in [12] that this class of semigroup control systems, with a sequence of approximating subspaces which satisfy assumption (H1), leads to a sequence of approximating control systems which satisfy assumptions (A1)-(A5) of the previous section. Furthermore, as in section (3.1) we can choose K_n, F_n so that $K_n \to K$ and $F_n \to F$.

Note that it is only required that projections onto X_n converge in the V-norm. A similar result in [8] requires an inverse approximation property:

$$\inf_{x \in X_n} \| R(s; A)z - x \|_V \leq \epsilon_1(n) \| z \|_X$$

$$\inf_{x \in X_n} \| R(s; A^*)z - x \|_V \leq \epsilon_2(n) \| z \|_X$$

where $\epsilon_1(N), \epsilon_2(N) \to 0$ as $N \to \infty$. These conditions are stronger, and more difficult to verify than (H1). Furthermore, the above conditions can only be satisfied if $R(s; A)$ is compact. Problems such as control of structural vibrations with Kelvin-Voigt damping are thus excluded.

4 Controller Convergence

It will now be shown that if both (A4) and (A5) hold in addition to the usual (A1)-(A3), then for large enough model order, the finite-dimensional controllers stabilize the semigroup control system (1). If a stronger version of (A4) and (A5) holds, then the controllers converge uniformly (in $M(\ \mathcal{H}_\infty\ )$) to an infinite-dimensional controller, and the closed loop performance obtained with the finite-dimensional controllers converges to the closed loop performance obtained with the infinite-dimensional controller.

Theorem 4.1 *Assume the semigroup control system (1) is jointly stabilizable/detectable and that the approximations used in design of the finite-dimensional controllers (3) satisfy assumptions (A1)-(A5). Denote the transfer functions of the controllers by H_n and the transfer function of (1) by G. Assume, in addition, that the feedback operators K_n converge strongly to an operator $K \in \mathcal{B}(X, U)$:*

$$\lim_{n \to \infty} K_n P_n x = K x, \quad x \in X.$$

Then, for sufficiently large n, H_n stabilizes the infinite-dimensional system G. For such n, the operators

$$A_{on} = \begin{bmatrix} A & -BK_n \\ F_n C & A_n - B_n K_n - F_n C_n \end{bmatrix} \tag{17}$$

generate stable semigroups.

Proof: Define

$$X_n(s) = K_n R(s, A_n - F_n C_n) F_n$$

and

$$Y_n(s) = I + K_n R(s, A_n - F_n C_n) B_n.$$

It is well-known (eg. [19]) that (X_n, Y_n) is a l.c.f. for the finite-dimensional controller H_n, and that

$$X_n N_n + Y_n D_n = I$$

where (N_n, D_n) is the r.c.f of G_n defined by (9) and (10) using the stabilizing feedback operator K_n.

It is clear from the assumptions that

$$\| [X_n\ Y_n] \|_\infty \leq M$$

for some constant M. Let (N, D) be the r.c.f. of G defined by (9) and (10) using the stabilizing feedback controller K. It was shown in [12] that $\lim_{n \to \infty} N_n = N$ and $\lim_{n \to \infty} D_n = D$

11

and so, for sufficiently large n, say $n > n_o$,

$$\| [X_n \ Y_n] \begin{bmatrix} N - N_n \\ D - D_n \end{bmatrix} \|_\infty < 1, \qquad n > n_o.$$

It follows that

$$U_n = X_n N + Y_n D$$

has an inverse in $M(\ \mathcal{H}_\infty\ )$ for $n > n_o$.

The closed loop matrix $\Delta(G, H_n)$ can be written in terms of $G = ND^{-1}$ and $H_n = Y_n^{-1} X_n$ as

$$\Delta(G, H_n) = \begin{bmatrix} I - NU_n^{-1}X_n & -NU_n^{-1}Y_n \\ DU_n^{-1}X_n & DU_n^{-1}Y_n \end{bmatrix}.$$

Since $\Delta(G, H_n) \in M(\ \mathcal{H}_\infty\ )$ the closed loop system is, by definition input/output stable. It is trivial to show that A_{on} is the infinitesimal generator for the closed loop system. Internal stability of the closed loop follows from the fact that the controller is stable and Theorem 2.3. Hence, A_{on} generates an exponentially stable semigroup. Alternatively, it can be shown directly that the closed loop system is stabilizable and detectable and Theorem 2.1 applied. □

The dual situation, where K_n is bounded and F_n converges strongly to an operator F, can be shown identically, using convergence of a sequence of l.c.f's of the systems G_n and boundedness of r.c.f.'s of the compensators H_n. We also have the following:

Corollary 4.2 *Let α be as in assumptions (A4)-(A5) and let σ be any real number, $\alpha > \sigma \geq 0$. For n sufficiently large, the operator A_{on} (17) generates an σ-stable semigroup.*

Proof: As a consequence of assumptions (A4) and (A5) we can replace input/output stability and internal stability by input/output σ-stability and internal σ-stability in the previous theorem. The algebra $\mathcal{H}_\infty$ is replaced by the shifted algebra $H_{\infty\sigma}$. □

Theorem 4.3 *Assume that the approximation scheme satisfies assumptions (A1)-(A5) and that the compensator design sequence is convergent:*

$$\lim_{n \to \infty} K_n P_n x = K x,$$

for all $x \in X$ and

$$\lim_{n \to \infty} F_n y = F y$$

for all $y \in R^p$. Let H denote the transfer function of the infinite-dimensional controller whose state-space realization is

$$\dot{z}(t) = (A - BK)z(t) + F(u(t) - Cz(t)) \tag{18}$$

$$y(t) = Kz(t).$$

The controller transfer functions H_n converge in $M(\mathcal{H}_\infty)$ (i.e. uniformly in s) to H.

Proof: We first show that l.c.f. 's of the controller converge:

$$\lim_{n \to \infty} \| X - X_n \|_\infty = 0, \qquad \lim_{n \to \infty} \| Y - Y_n \|_\infty = 0.$$

Define X_n, Y_n as in Theorem 4.1 . Denoting the semigroup generated by $A_n - F_n C_n$ by $S_{Fn}(t)$, the time-domain representations of these transfer functions are

$$X_n(t) := K_n S_{Fn}(t) F_n,$$

$$Y_n(t) := \delta(t) I + K_n S_{Fn}(t) B_n$$

where $\delta(t)$ denotes the Dirac delta distribution. Define

$$X(t) := K S_F(t) F,$$

$$Y(t) := \delta(t) I + K S_F(t) B.$$

The proof is identical to that used to show convergence of coprime factors of the approximating systems in [12]. Due to convergence of K_n to K, and F_n to F, we can show that,

$$\lim_{n \to \infty} \int_0^\infty \| X_n(t) - X(t) \|_{R^{m+p}} dt = 0,$$

$$\lim_{n \to \infty} \int_0^\infty \| Y_n(t) - Y(t) \|_{R^{m+p}} dt = 0$$

where R^{m+p} indicates some norm on operators from R^m to R^p. Convergence in this norm implies convergence of the Laplace transforms of these functions in $M(\mathcal{H}_\infty)$. Each controller H_n and the infinite-dimensional controller H are stable (elements of $M(\mathcal{H}_\infty)$). This implies that, Y_n, Y are invertible in $M(\mathcal{H}_\infty)$ and since $\lim_{n\to\infty} \| Y_n - Y \|_\infty = 0$,

$$\lim_{n \to \infty} \| Y_n^{-1} - Y^{-1} \|_\infty = 0$$

and H_n converges to H in norm. (This reflects the fact that the uniform topology is the restriction of the graph topology to stable systems.) Since (X, Y) is a l.c.f. for the infinite-dimensional controller H, the result follows. $\square$

Note that the controllers actually converge in the stronger norm:

$$\lim_{n \to \infty} \int_0^\infty \| H_n(t) - H(t) \|_{R^{m+p}} e^{\gamma t} dt = 0,$$

where $\alpha > \gamma > 0$ (α as in (A4) and (A5)) although this is not used here.

13

Corollary 4.4 *The closed loop operator $\Delta(G_n, H_n)$ converges in $M(\mathcal{H}_\infty)$ to $\Delta(G, H)$.*

We can write the closed loop operator $\Delta(G, H)$ in terms of stable coprime factors as

$$\Delta(G, H) = \begin{bmatrix} I - NX & -NY \\ DX & DY \end{bmatrix}$$

and similarly,

$$\Delta(G_n, H_n) = \begin{bmatrix} I - N_n X_n & -N_n Y_n \\ D_n X_n & D_n Y_n \end{bmatrix}.$$

The result now follows from convergence of the coprime factors. $\square$

Thus we have shown uniform convergence of the controllers to an infinite-dimensional controller, and similar convergence of the closed-loop systems.

4.1 LQG Optimal Control

Commonly, a compensator of the form (18) is designed in two steps, through construction of an optimal regulator and an optimal detector. Assume that (1) is jointly stabilizable/detectable, so that construction of an internally stable closed loop is possible.

If Q is a self-adjoint positive semi-definite operator and R is self-adjoint positive definite, then the optimal solution of

$$J(u) = \int_0^\infty \langle Qx(t), x(t) \rangle_X + \langle Ru(t), u(t) \rangle dt$$

is given by the feedback $u(t) = -Kx(t)$ where $K := B^* R^{-1} \Pi$ and Π is the unique, nonnegative, self-adjoint solution of the algebraic Riccati equation,

$$(A^* \Pi + \Pi A - \Pi B R^{-1} B^* \Pi + Q)z = 0, \qquad \text{for all} \quad z \in D(A). \tag{19}$$

The operator $A - BK$ generates a stable semigroup on X.

Similarly, we may obtain the observer gain F by solving the dual problem

$$A\Sigma + \Sigma A^* - \Sigma C^* R_o^{-1} C\Sigma + Q_o = 0 \tag{20}$$

where Q_o is self-adjoint positive semi-definite and R_o is self-adjoint positive definite. Setting $F = \Sigma C^* R_o^{-1}$ where Σ is the unique, nonnegative, self-adjoint solution to the above Riccati equation, $A - FC$ generates a stable semigroup on X.

Since the above infinite-dimensional Riccati equations cannot be solved exactly, approximate solutions are calculated by using the approximations A_n, B_n, C_n. Thus, if the approximations are stabilizable, we obtain $K_n = R^{-1} B_n^* \Pi_n$ where Π_n solves, for $Q_n = P_n Q P_n$,

$$A_n^* \Pi_n + \Pi_n A_n - \Pi_n B_n R^{-1} B_n^* \Pi_n + Q_n = 0. \tag{21}$$

14

Similarly, if the approximations are detectable, the operators $F_n = \Sigma_n \, C_n^* \, R_o^{-1}$ are found by solving, for $Q_{on} = P_n Q_o P_n$,

$$A_n \Sigma_n + \Sigma_n \, A_n^* - \Sigma_n \, C_n^* \, R_o^{-1} C_n \Sigma_n + Q_{on} = 0. \tag{22}$$

The theory on convergence of solutions of approximating Riccati equations is fairly complete, and we can state the following which is a consequence of [3] and the theorems in the previous section. Convergence of the solutions requires that the adjoint semigroups $S_n^*(t)$ converge strongly, uniformly on bounded intervals, to $S^*(t)$. This assumption, together with (A1)-(A5) implies the stronger versions of (A4) and (A5) required for Theorems 4.1-4.3.

Theorem 4.5 *Assume that the approximation scheme satisfies the usual (A1)-(A3) and that it is uniformly stabilizable (A4) and uniformly detectable (A5). Assume also, that for every $x \in X$, the adjoint semigroups $S_n^*(t)$ converge strongly, uniformly on bounded intervals, to $S^*(t)$:*

$$\lim_{n \to \infty} \sup_{0 \le t \le T} \| \; S_n^*(t) P_n x - P_n \, S^*(t) x \| = 0, \qquad T \ge 0. \tag{23}$$

Then:

1. *The controller (3) obtained by solving the finite-dimensional ARE's (21) and (22) to obtain K_n, F_n stabilizes the infinite-dimensional system for all sufficiently large n.*

2. *This closed loop system is internally σ-stable, for any $\sigma < \alpha$.*

3. *The controller transfer functions converge uniformly (i.e. in $M(\mathcal{H}_\infty)$) to the transfer function of the optimal regulator/observer (18) obtained by solving the infinite-dimensional ARE's.*

4. *The closed loop operator $\Delta(G_n, H_n)$ converges uniformly to $\Delta(G, H)$.*

Proof: It has been shown [3] that these assumptions are sufficient to guarantee that $\lim_{n \to \infty} K_n x = Kx$ and that $\lim_{n \to \infty} F_n y = Fy$ and that these feedback operators yield uniformly stable semigroups : (A4) and (A5) are satisfied. The conclusions now follow immediately from Theorems 4.1-4.3 . □

5 Conclusions

A similar result to Theorem 3.1 and Corollary 3.2 has been shown in [8]. However, the use of coprime factorizations leads to a considerably shorter proof. Also, the results in [8] depend on several assumptions that are not required here. Unlike [8], the spectrum determined growth assumption not required at any point. Furthermore, both classes of approximation schemes discussed in [8] have the property that the resolvants of the approximations converge in norm to the original resolvant:

$$\lim_{n \to \infty} \| R(\lambda, A_n) - R(\lambda, A)\| = 0$$

and this property is used in the proofs. This restricts the class of systems under consideration to those with compact resolvants, and the approximation scheme to those with this convergence property. As demonstrated here, these assumptions are not necessary.

The results here showing uniform convergence of a sequence of controllers, and of the corresponding closed-loop systems, have not been previously obtained. Gibson and Adamian [7] study the problem of convergence of LQG controller design for flexible structures, when the controller is designed using approximations to the solution of the partial differential equation describing the vibrations. Numerical results in [7], for finite-element approximations to structural vibrations indicated convergence of the controller transfer functions, but the authors were unable to prove Theorem (4.3) in the general case.

One important consequence of convergence of the controller transfer functions, is that it justifies the use of such order reduction methods as Hankel norm reductions, balanced truncations, and coprime factor reduction. The order of the approximation (2) could be increased until the controllers (3) have converged sufficiently in $M(\ \mathcal{H}_\infty\ )$. The numerical results in [7] for control of structural vibrations indicate that at least in some cases, convergence is obtained fairly quickly. The controller order can then be reduced using a standard technique such as balanced realizations.

References

[1] Balas, M.J., "Finite-Dimensional Control of Distributed Parameter Systems by Galerkin Approximation" *J. Math. Anal. Appl.* , Vol 114, pg 17-36, 1986.

[2] Banks, H.T. and J.A. Burns, " Hereditary Control Problems: Numerical Methods based on Averaging Approximations", *SIAM Jour. on Control and Optimization*, Vol 16, pg 169-208, 1978

[3] Banks, H.T. and Kunisch, K., "The Linear Regulator Problem for Parabolic Systems", *SIAM Jour. on Control and Optimization*, Vol 22, pg 684-698, 198 4.

[4] Burns, J., Cliff, E.M, Powers, R.K., "Computational Methods for the Control of Distributed Parameter Systems", *ICASE*, NASA Langley Research Center, , # 86-32, 1986.

[5] Callier, F.M. and Desoer, C.A., "An Algebra of Transfer Functions of Linear Distributed Linear TIme-invariant Systems." *IEEE Trans. on Automatic Control*, Vol AC-25, pg 651-662, 1978.

[6] Curtain, Ruth F., "Robust Stabilizability of Normalized Coprime Factors: the Infinite Dimensional Case", TW-291, University of Groningen, Groningen, The Netherlands.

[7] Gibson, J.S. and Adamian A., "Approximation Theory for Linear-Quadratic-Gaussian Optimal Control of Flexible Structures" *SIAM Jour. Control & Optimization*, Vol 29, No 1, pg 1-37, Jan 1991.

[8] Ito, K., "Finite-Dimensional Compensators for Infinite-Dimensional Systems via Galerkin-Type Approximation" *SIAM J. Control and Optimization*, Vol 28, No 6, pg 1251-1269, November 1990.

[9] Ito, K., " Strong Convergence and Convergence Rates of Approximating Solutions for Algebraic Riccati Equations in Hilbert Spaces" Proceedings of the 3rd International Conference on Distributed Parameter Systems, Springer-Verlag, 1986.

[10] Jacobson, C.A. and Nett, C.N., "Linear State-Space Systems in Infinite-Dimensional Space: The Role and Characterization of Joint Stabilizability/Detectability", *IEEE Trans. on Automatic Control*, Vol AC-33, pg 541-549, 1988.

[11] Kato, T., "Perturbation Theory for Linear Operators" Springer-Verlag Inc., 1966.

[12] Morris, K.A., "Design of Finite-Dimensional Controllers for Infinite-Dimensional Systems by Approximation", submitted, *SIAM Jour. on Control & Optimization*.

[13] Nett, C. N., Jacobson, C.A. and Balas, M.J., "Fractional Representation Theory: Robustness with Applications to Finite-dimensional Control of a Class of Linear Distributed Systems", *Proc. of the 22nd IEEE CDC*, Vol. 1, IEEE Control SYstems Society, 1983, pg. 268-280.

[14] Pazy, A., "Semigroups of Linear Operators and Applications to Partial Differential Equations", Springer-Verlag, 1983.

[15] Salamon, Dietmar, "Structure and Stability of Finite Dimensional Approximations for Functional Differential Equations", *SIAM Jour. on Control and Optimization*, Vol 23, pg 928-951, 1985.

[16] Smith, M.C., "On Stabilization and the Existence of Coprime Factorizations", *IEEE Trans. on Automatic Control*, Vol AC-34, pg 1005-1007, 1989.

[17] Showalter, R.E.,"Hilbert Space Methods for Partial Differential Equations" Pitman Publishing Ltd, London, 1977.

[18] Vidyasagar, M., "The Graph Metric for Unstable Plants and Robustness Estimates for Feedback Stability" *IEEE Trans. on Automatic Control*, Vol AC-29, 1984.

[19] Vidyasagar, M., *Control System Synthesis: A Factorization Approach*, MIT Press, Cambridge, 1985.

[20] Zhu, S.Q.,"Graph Topology and Gap Topology for Unstable Systems" *IEEE Trans. on Automatic Control*, Vol AC-34, pg 848-855, 1989.

REPORT DOCUMENTATION PAGE

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.

1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE December 1991	3. REPORT TYPE AND DATES COVERED Contractor Report

4. TITLE AND SUBTITLE

CONVERGENCE OF CONTROLLERS DESIGNED USING STATE SPACE METHODS

5. FUNDING NUMBERS

C NAS1-18605

WU 505-90-52-01

6. AUTHOR(S)

K. A. Morris

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

Institute for Computer Applications in Science
and Engineering
Mail Stop 132C, NASA Langley Research Center
Hampton, VA 23665-5225

8. PERFORMING ORGANIZATION REPORT NUMBER

ICASE Report No. 91-85

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

National Aeronautics and Space Administration
Langley Research Center
Hampton, VA 23665-5225

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

NASA CR-189575
ICASE Report No. 91-85

11. SUPPLEMENTARY NOTES

Langley Technical Monitor: Michael F. Card
Final Report

Submitted to SIAM Journal
on Control & Optimization

12a. DISTRIBUTION/AVAILABILITY STATEMENT

Unclassified-Unlimited

Subject Categories 63, 64

12b. DISTRIBUTION CODE

13. ABSTRACT (Maximum 200 words)

In this paper convergence of finite-dimensional controllers for infinite-dimensional systems designed using approximations is examined. Stable coprime factorization theory is used to show that under the standard assumptions of uniform stabilizability/detectability, the controllers stabilize the original system for large enough model order. The controllers converge uniformly to an infinite-dimensional controller, as does the closed loop response.

14. SUBJECT TERMS

infinite-dimensional systems; finite-dimensional controllers;
graph topology; coprime factorizations

15. NUMBER OF PAGES

20

16. PRICE CODE

A03

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified		